

LAMPLIGHTER GROUPS AND VON NEUMANN'S CONTINUOUS REGULAR RING

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ABSTRACT. Let Γ be a discrete group. Following Linnell and Schick one can define a continuous ring $c(\Gamma)$ associated with Γ . They proved that if the Atiyah Conjecture holds for a torsion-free group Γ , then $c(\Gamma)$ is a skew field. Also, if Γ has torsion and the Strong Atiyah Conjecture holds for Γ , then $c(\Gamma)$ is a matrix ring over a skew field. The simplest example when the Strong Atiyah Conjecture fails is the lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$. It is known that $\mathbb{C}(\mathbb{Z}_2 \wr \mathbb{Z})$ does not even have a classical ring of quotients. Our main result is that if H is amenable, then $c(\mathbb{Z}_2 \wr H)$ is isomorphic to a continuous ring constructed by John von Neumann in the 1930's.

Keywords. continuous rings, von Neumann algebras, the algebra of affiliated operators, lamplighter group

1. INTRODUCTION

Let us consider $\text{Mat}_{k \times k}(\mathbb{C})$ the algebra of k by k matrices over the complex field. This ring is a unital $*$ -algebra with respect to the conjugate transposes. For each element $A \in \text{Mat}_{k \times k}(\mathbb{C})$ one can define A^* satisfying the following properties.

- $(\lambda A)^* = \bar{\lambda} A^*$
- $(A + B)^* = A^* + B^*$
- $(AB)^* = B^* A^*$
- $0^* = 0, 1^* = 1$

Also, each element has a normalized rank $\text{rk}(A) = \text{Rank}(A)/k$ with the following properties.

- $\text{rk}(0) = 0, \text{rk}(1) = 1,$
- $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$
- $\text{rk}(AB) \leq \min\{\text{rk}(A), \text{rk}(B)\}$
- $\text{rk}(A^*) = \text{rk}(A)$
- If e and f are orthogonal idempotents then $\text{rk}(e + f) = \text{rk}(e) + \text{rk}(f)$.

The ring $\text{Mat}_{k \times k}(\mathbb{C})$ has an algebraic property namely, von Neumann called regularity: Any principal left-(or right) ideal can be generated by an idempotent. Furthermore, among these generating idempotents there is a unique projection (that is $\text{Mat}_{k \times k}(\mathbb{C})$ is a $*$ -regular ring). In a von Neumann regular ring any non-zero-divisor is necessarily invertible. One can also observe that the algebra of matrices is proper, that is $\sum_{i=1}^n a_i a_i^* = 0$ implies that all the matrices a_i are zero. One should note that if R is a $*$ -regular ring with a rank

AMS SUBJECT CLASSIFICATION: 16A30, 46L10 RESEARCH PARTLY SPONSORED BY MTA RENYI "LENDULET" GROUPS AND GRAPHS RESEARCH GROUP

function, then the rank extends to $\text{Mat}_{k \times k}(R)$ [6], where the extended rank has the same property as rk except that the rank of the identity is k .

One can immediately see that the rank function defines a metric $d(A, B) := \text{rk}(A - B)$ on any algebra with a rank, and the matrix algebra is complete with respect to this metric. These complete $*$ -regular algebras are called continuous $*$ -algebras (see [5] for an extensive study of continuous rings). Note that for the matrix algebras the possible values of the rank functions are $0, 1/k, 2/k, \dots, 1$. John von Neumann observed that there are some interesting examples of infinite dimensional continuous $*$ -algebras, where the rank function can take any real values in between 0 and 1. His first example was purely algebraic.

Example 1. Let us consider the following sequence of diagonal embeddings.

$$\mathbb{C} \rightarrow \text{Mat}_{2 \times 2}(\mathbb{C}) \rightarrow \text{Mat}_{4 \times 4}(\mathbb{C}) \rightarrow \text{Mat}_{8 \times 8}(\mathbb{C}) \rightarrow \dots$$

One can observe that all the embeddings are preserving the rank and the $*$ -operation. Hence the direct limit $\varinjlim \text{Mat}_{2^k \times 2^k}(\mathbb{C})$ is a $*$ -regular ring with a proper rank function. The addition, multiplication, the $*$ -operation and the rank function can be extended to the metric completion \mathcal{M} of the direct limit ring. The resulting algebra \mathcal{M} is a simple, proper, continuous $*$ -algebra, where the rank function can take all the values on the unit interval.

Example 2. Consider a finite, tracial von Neumann algebra \mathcal{N} with trace function $\text{tr}_{\mathcal{N}}$. Then \mathcal{N} is a $*$ -algebra equipped with a rank function. If P is a projection, then $\text{rk}_{\mathcal{N}}(P) = \text{tr}_{\mathcal{N}}(P)$. For a general element $A \in \mathcal{N}$, $\text{rk}_{\mathcal{N}}(A) = 1 - \lim_{t \rightarrow \infty} \int_0^t \text{tr}_{\mathcal{N}}(E_{\lambda}) d\lambda$, where $\int_0^{\infty} E_{\lambda} d\lambda$ is the spectral decomposition of A^*A . In general, \mathcal{N} is not regular, but it has the Ore property with respect to its zero divisors. The Ore localization of \mathcal{N} with respect to its non-zero-divisors is called the algebra of affiliated operators and denoted by $U(\mathcal{N})$. These algebras are also proper continuous $*$ -algebras [1]. The rank of an element $A \in U(\mathcal{N})$ is given by the trace of the projection generating the principal ideal $U(\mathcal{N})A$. It is important to note, that $U(\mathcal{N})$ is the rank completion of \mathcal{N} (Lemma 2.2 ([12])).

Linnell and Schick observed [9] that if X is a subset of a proper $*$ -regular algebra R , then there exists a smallest $*$ -regular subalgebra containing X , the $*$ -regular closure. Now let Γ be a countable group and $\mathbb{C}\Gamma$ be its complex group algebra. Then one can consider the natural embedding of the group algebra to its group von Neumann algebra $\mathbb{C}\Gamma \rightarrow \mathcal{N}\Gamma$. Let $U(\Gamma)$ denote the Ore localization of $\mathcal{N}(\Gamma)$ and the embedding $\mathbb{C}\Gamma \rightarrow U(\Gamma)$. Since $U(\Gamma)$ is a proper $*$ -regular ring, one can consider the smallest $*$ -algebra $\mathcal{A}(\Gamma)$ in $U(\Gamma)$ containing $\mathbb{C}(\Gamma)$. Let $c(\Gamma)$ be the completion of the algebra \mathcal{A} above. It is a continuous $*$ -algebra [5]. Of course, if the rank function has only finitely many values in \mathcal{A} , then $c(\Gamma)$ equals to $\mathcal{A}(\Gamma)$. Note that if $\mathbb{C}\Gamma$ is embedded into a continuous $*$ -algebra T , then one can still define $c_T(\Gamma)$ as the smallest continuous ring containing $\mathbb{C}\Gamma$. In [3] we proved that if Γ is amenable, $c(\Gamma) = c_T(\Gamma)$ for any embedding $\mathbb{C}\Gamma \rightarrow T$ associated to sofic representations of Γ , hence $c(\Gamma)$ can be viewed as a canonical object. Linnell and Schick calculated the algebra $c(\Gamma)$ for several groups, where the rank function has only finitely many values on \mathcal{A} . They proved the following results:

- If Γ is torsion-free and the Atiyah Conjecture holds for Γ , then $c(\Gamma)$ is a skew-field. This is the case, when Γ is amenable and $\mathbb{C}\Gamma$ is a domain. Then $c(\Gamma)$ is the Ore localization of $\mathbb{C}\Gamma$. If Γ is the free group of k generators, then $c(\Gamma)$ is the Cohen-Amitsur free skew field of k generators. The Atiyah Conjecture for a torsion-free group means that the rank of an element in $\text{Mat}_{k \times k}(\mathbb{C}\Gamma) \subset \text{Mat}_{k \times k}(U(\mathcal{N}(\Gamma)))$ is an integer.
- If the orders of the finite subgroups of Γ are bounded and the Strong Atiyah Conjecture holds for Γ , then $c(\Gamma)$ is a finite dimensional matrix ring over some skew field. In this case the Strong Atiyah Conjecture means that the ranks of an element in $\text{Mat}_{k \times k}(\mathbb{C}\Gamma) \subset \text{Mat}_{k \times k}(U(\mathcal{N}(\Gamma)))$ is in the abelian group $\frac{1}{\text{lcm}(\Gamma)}\mathbb{Z}$, where $\text{lcm}(\Gamma)$ indicates the least common multiple of the orders of the finite subgroups of Γ .

The lamplighter group $\Gamma = \mathbb{Z}_2 \wr \mathbb{Z}$ has finite subgroups of arbitrarily large orders. Also, although Γ is amenable, $\mathbb{C}\Gamma$ does not satisfy the Ore condition with respect to its non-zero-divisors [8]. In other words, it has no classical ring of quotients. The goal of this paper is to calculate $c(\mathbb{Z}_2 \wr \mathbb{Z})$ and even $c(\mathbb{Z}_2 \wr H)$, where H is a countably infinite amenable group.

Theorem 1. *If H is a countably infinite amenable group, then $c(\mathbb{Z}_2 \wr H)$ is the simple continuous ring \mathcal{M} of von Neumann.*

2. CROSSED PRODUCT ALGEBRAS

In this section we recall the notion of crossed product algebras and the group-measure space construction of Murray and von Neumann. Let \mathcal{A} be a unital, commutative $*$ -algebra and $\phi : \Gamma \rightarrow \text{Aut}(\mathcal{A})$ be a representation of the countable group Γ by $*$ -automorphisms. The associated crossed product algebra $\mathcal{A} \rtimes \Gamma$ is defined the following way. The elements of $\mathcal{A} \rtimes \Gamma$ are the finite formal sums

$$\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma,$$

where $a_\gamma \in \mathcal{A}$. The multiplicative structure is given by

$$\delta \cdot a_\gamma = \phi(\delta)(a_\gamma) \cdot \delta.$$

The $*$ -structure is defined by $\gamma^* = \gamma^{-1}$ and $(\gamma \cdot a)^* = a^* \cdot \gamma^{-1}$. Note that

$$(\delta \cdot a_\gamma)^* = (\phi(\delta)a_\gamma \cdot \delta)^* = \delta^* \cdot \phi(\delta)a_\gamma^* = \phi(\delta^{-1})\phi(\delta)a_\gamma^* \cdot \delta^{-1} = a_\gamma^* \cdot \delta^*.$$

Now let (X, μ) be a probability measure space and $\tau : \Gamma \curvearrowright X$ be a measure preserving action of a countable group Γ on X . Then we have a $*$ -representation $\hat{\tau}$ of Γ in $\text{Aut}(L^\infty(X, \mu))$, where $L^\infty(X, \mu)$ is the commutative $*$ -algebra of bounded measurable functions on X (modulo zero measure perturbations).

$$\hat{\tau}(\gamma)(f)(x) = f(\tau(\gamma^{-1})(x)).$$

Let $\mathcal{H} = l^2(\Gamma, L^2(X, \mu))$ be the Hilbert-space of $L^2(X, \mu)$ -valued functions on Γ . That is, each element of \mathcal{H} can be written in the form of

$$\sum_{\gamma \in \Gamma} b_\gamma \cdot \gamma,$$

where $\sum_{\gamma \in \Gamma} \|b_\gamma\|^2 < \infty$. Then we have a representation L of $L^\infty(X, \mu) \rtimes \Gamma$ on $l^2(\Gamma, L^2(X, \mu))$ by

$$L\left(\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma\right)\left(\sum_{\delta \in \Gamma} b_\delta \cdot \delta\right) = \sum_{\delta \in \Gamma} \left(\sum_{\gamma \in \Gamma} a_\gamma(\hat{\tau}(\gamma)(\beta_\delta)) \cdot \gamma\delta\right).$$

Note that $L(\sum_{\gamma \in \Gamma} a_\gamma \cdot \gamma)$ is always a bounded operator. A trace is given on $L^\infty(X, \mu) \rtimes \Gamma$ by

$$\text{Tr}(S) = \int_X a_1(x) d\mu(x).$$

The weak operator closure of $L(L_c^\infty(X, \mu) \rtimes \Gamma)$ in $B(l^2(\Gamma, L^2(X, \mu)))$ is the von Neumann algebra $\mathcal{N}(\tau)$ associated to the action. Here $L_c^\infty(X, \mu)$ denotes the subspace of functions in $L^\infty(X, \mu)$ having only countable many values.

Note that one can extend Tr to $\text{Tr}_{\mathcal{N}(\tau)}$ on the von Neumann algebra to make it a tracial von Neumann algebra.

We will denote by $c(\tau)$ the smallest continuous algebra in $U(\mathcal{N}(\tau))$ containing $L_c^\infty(X, \mu) \rtimes \Gamma$. One should note that the weak closure of $L_c^\infty(X, \mu) \rtimes \Gamma$ in $B(l^2(\Gamma, L^2(X, \mu)))$ is the same as the weak closure of $L^\infty(X, \mu) \rtimes \Gamma$. Hence our definition for the von Neumann algebra of an action coincides with the classical definition. On the other hand, $c(L_c^\infty(X, \mu) \rtimes \Gamma)$ is smaller than $c(L^\infty(X, \mu) \rtimes \Gamma)$.

3. THE BERNOULLI ALGEBRA

Let H be a countable group. Consider the Bernoulli shift space $B_H := \prod_{h \in H} \{0, 1\}$ with the usual product measure ν_H . The probability measure preserving action $\tau_H : H \curvearrowright (B_H, \nu_H)$ is defined by

$$\tau_H(\delta)(x)(h) = x(\delta^{-1}h),$$

where $x \in B_H$, $\delta, h \in H$. Let \mathcal{A}_H be the commutative $*$ -algebra of functions that depend only on finitely many coordinates of the shift space. It is well-known that the Rademacher functions $\{R_S\}_{S \subset H, |S| < \infty}$ form a basis in \mathcal{A}_H , where

$$R_S(x) = \prod_{\delta \in S} \exp(i\pi x(\delta)).$$

The Rademacher functions with respect to the pointwise multiplication form an Abelian group isomorphic to $\oplus_{h \in H} \mathbb{Z}_2$ the Pontrjagin dual of the compact group B_H satisfying

- $R_S R_{S'} = R_{S \Delta S'}$
- $\int_{B_H} R_S d\nu = 0$, if $|S| > 0$
- $R_\emptyset = 1$.

The group H acts on \mathcal{A}_H by

$$\hat{\tau}_H(\delta)(f)(x) = f(\tau_H(\delta^{-1})(x)) .$$

Hence,

$$\hat{\tau}_H(\delta)R_S = R_{\delta S} .$$

Therefore, the elements of $\mathcal{A}_H \rtimes H$ can be uniquely written as in the form of the finite sums

$$\sum_{\delta} \sum_S c_{\delta,S} R_S \cdot \delta ,$$

where $\delta \cdot R_S = R_{\delta S} \cdot \delta$.

Now let us turn our attention to the group algebra $\mathbb{C}(\mathbb{Z}_2 \wr H)$. For $\delta \in H$, let t_δ be the generator in $\sum_{h \in H} \mathbb{Z}_2$ belonging to the δ -component. Any element of $\mathbb{C}(\mathbb{Z}_2 \wr H)$ can be written in a unique way as a finite sum

$$\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta ,$$

where $t_S = \prod_{s \in S} t_s$, $\delta \cdot t_S = t_{\delta S}$, $t_S t_{S'} = t_{S \Delta S'}$. Also note that

$$\text{Tr}(\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta) = c_{1,\emptyset} .$$

Hence we have the following proposition.

Proposition 3.1. *There exists a trace preserving $*$ -isomorphism $\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{A}_H \rtimes H$ such that*

$$\kappa(\sum_{\delta} \sum_S c_{\delta,S} t_S \cdot \delta) = \sum_{\delta} \sum_S c_{\delta,S} R_S \cdot \delta .$$

Recall that if $A \subset \mathcal{N}_1$, $B \subset \mathcal{N}_1$ are weakly dense $*$ -subalgebras in finite tracial von Neumann algebras \mathcal{N}_1 and \mathcal{N}_2 and $\kappa : A \rightarrow B$ is a trace preserving $*$ -homomorphism, then κ extends to a trace preserving isomorphism between the von Neumann algebras themselves (see e.g. [7] Corollary 7.1.9.). Therefore, $\kappa : \mathbb{C}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{A}_H \rtimes H$ extends to a trace (and hence rank) preserving isomorphism between the von Neumann algebras $\mathcal{N}(\mathbb{Z}_2 \wr H)$ and $\mathcal{N}(\tau_H)$.

Proposition 3.2. *For any countable group H ,*

$$c(\mathbb{Z}_2 \wr H) \cong c(\tau_H) .$$

Proof. The rank preserving isomorphism $\kappa : \mathcal{N}(\mathbb{Z}_2 \wr H) \rightarrow \mathcal{N}(\tau_H)$ extends to a rank preserving isomorphism between the rank completions, that is, the algebras of affiliated operators. It is enough to prove that the rank closure of $\mathcal{A}_H \rtimes H$ is $L_c^\infty(B_H, \nu_H) \rtimes H$.

Lemma 3.1. *Let $f \in L_c^\infty(B_H, \nu_H)$. Then $rk_{\mathcal{N}(\tau_H)}(f) = \nu_H(\text{supp}(f))$.*

Proof. By definition,

$$\mathrm{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \lim_{\lambda \rightarrow 0} \mathrm{tr}_{\mathcal{N}(\tau_H)} E_\lambda,$$

where E_λ is the spectral projection of f^*f corresponding to λ .

$$\mathrm{tr}_{\mathcal{N}(\tau_H)} E_\lambda = \nu_H(\{x \mid |f^2(x)| \leq \lambda\}).$$

Hence, $\mathrm{rk}_{\mathcal{N}(\tau_H)}(f) = 1 - \nu_H(\{x \mid f^2(x) = 0\}) = \nu_H(\mathrm{supp}(f))$. \square

Let $\{m_n\}_{n=1}^\infty \subset \mathcal{A}_H, m_n \xrightarrow{\mathrm{rk}} m \in L_c^\infty(B_H, \nu_H)$. Then $m_n \cdot \gamma \xrightarrow{\mathrm{rk}} m \cdot \gamma$. Therefore our proposition follows from the lemma below.

Lemma 3.2. \mathcal{A}_H is dense in $L_c^\infty(B_H, \nu_H)$ with respect to the rank metric.

Proof. By Lemma 3.1, $L_{fin}^\infty(B_H, \nu_H)$ is dense in $L_c^\infty(B_H, \nu_H)$, where $L_{fin}^\infty(B_H, \nu_H)$ is the $*$ -algebra of functions taking only finitely many values. Recall that $V \subset B_H$ is a basic set if $1_V \in \mathcal{A}_H$. It is well-known that any measurable set in B_H can be approximated by basic sets, that is for any $U \subset B_H$, there exists a sequence of basic sets $\{V_n\}_{n=1}^\infty$ such that

$$(1) \quad \lim_{n \rightarrow \infty} \nu_H(V_n \Delta U) = 0.$$

By (1) and Lemma 3.1

$$\lim_{n \rightarrow \infty} \mathrm{rk}_{\mathcal{N}(\tau_n)}(1_{V_n} - 1_U) = 0.$$

Let $f = \sum_{m=1}^l c_m 1_{U_m}$, where U_m are disjoint measurable sets. Let $\lim_{n \rightarrow \infty} \nu_H(V_n^m \Delta U_m) = 0$, where $\{V_n^m\}_{n=1}^\infty$ are basic sets. Then

$$\lim_{n \rightarrow \infty} \mathrm{rk}_{\mathcal{N}(\tau_n)}\left(\sum_{m=1}^l c_m 1_{V_n^m} - f\right) = 0.$$

Therefore, \mathcal{A}_H is dense in $L_{fin}^\infty(B_H, \nu_H)$. \square

4. THE ODOMETER ALGEBRA

The Odometer Algebra is constructed via the odometer action using the algebraic crossed product construction. Let us consider the compact group of 2-adic integers $\hat{\mathbb{Z}}_{(2)}$. Recall that $\hat{\mathbb{Z}}_{(2)}$ is the completion of the integers with respect to the dyadic metric

$$d_{(2)}(n, m) = 2^{-k},$$

where k is the power of two in the prime factor decomposition of $|m - n|$. The group $\hat{\mathbb{Z}}_{(2)}$ can be identified with the compact group of one way infinite sequences with respect to the binary addition.

The Haar-measure μ_{haar} on $\hat{\mathbb{Z}}_{(2)}$ is defined by $\mu_{\mathrm{haar}}(U_n^l) = 1/2^n$, where $0 \leq l \leq 2^n - 1$ and U_n^l is the clopen subset of elements in $\hat{\mathbb{Z}}_{(2)}$ having residue l modulo 2^n . Let T be the addition map $x \rightarrow x + 1$ in $\hat{\mathbb{Z}}_{(2)}$. The map T defines an action $\rho : \mathbb{Z} \curvearrowright (\hat{\mathbb{Z}}_{(2)}, \mu_{\mathrm{haar}})$. The dynamical system $(T, \hat{\mathbb{Z}}_{(2)}, \mu_{\mathrm{haar}})$ is called the odometer action. As in Section 3, we

consider the $*$ -subalgebra of function \mathcal{A}_M in $L^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}})$ that depend only on finitely many coordinates of $\hat{\mathbb{Z}}_{(2)}$. We consider a basis for \mathcal{A}_M . For $n \geq 0$ and $0 \leq l \leq 2^n - 1$ let

$$F_n^l(x) = \exp\left(\frac{2\pi i x(\text{mod } 2^n)}{2^n} l\right).$$

Notice that $F_{n+1}^{2l} = F_n^l$. Then the functions $\{F_n^l\}_{n, l|(l,n)=1}$ form the Prüfer 2-group

$$\mathbb{Z}_{(2)} = \mathbb{Z}_1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8 \subset \dots$$

with respect to the pointwise multiplication. The discrete group $\mathbb{Z}_{(2)}$ is the Pontrjagin dual of the compact Abelian group $\hat{\mathbb{Z}}_{(2)}$. The element F_n^1 is the generator of the cyclic subgroup \mathbb{Z}_{2^n} . Note that

$$\int_{\hat{\mathbb{Z}}_{(2)}} F_n^l d\mu_{\text{haar}} = 0$$

except if $l = 0, n = 0$, when $F_n^l \equiv 1$. Observe that if $k \in \mathbb{Z}$ then

$$(2) \quad \rho(k)F_n^l = F_n^{l+k(\text{mod } 2^n)}$$

since $F_n^l(x - k) = F_n^{l+k(\text{mod } 2^n)}(x)$. Hence we have the following lemma.

Lemma 4.1. *The elements of $\mathcal{A}_M \rtimes \mathbb{Z}$ can be uniquely written as finite sums in the form*

$$\sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{n,l,k} F_n^l \cdot k,$$

where $k \cdot F_n^l = F_n^{l+k(\text{mod } 2^n)}$ and $F_0^0 = 1$.

5. PERIODIC OPERATORS

Definition 5.1. *A function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ is a periodic operator if there exists some $n \geq 1$ such that*

- $A(x, y) = 0$, if $|x - y| > 2^n$
- $A(x, y) = A(x + 2^n, y + 2^n)$.

Observe that the periodic operators form a $*$ -algebra, where

- $(A + B)(x, y) = A(x, y) + B(x, y)$
- $AB(x, y) = \sum_{z \in \mathbb{Z}} A(x, z)B(z, y)$
- $A^*(x, y) = A(y, x)$

Proposition 5.1. *The algebra of periodic operators \mathcal{P} is $*$ -isomorphic to a dense subalgebra of \mathcal{M} .*

Proof. We call $A \in \mathcal{P}$ an element of type- n if

- $A(x, y) = A(x + 2^n, y + 2^n)$
- $A(x, y) = 0$ if $0 \leq x \leq 2^n - 1, y > 2^n - 1$
- $A(x, y) = 0$ if $0 \leq x \leq 2^n - 1, y < 0$.

Clearly, the elements of type- n form an algebra \mathcal{P}_n isomorphic to $\text{Mat}_{2^n \times 2^n}(\mathbb{C})$ and $\mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$ is the diagonal embedding. Hence, we can identify the algebra of finite type elements $\mathcal{P}_f = \cup_{n=1}^{\infty} \mathcal{P}_n$ with $\varinjlim \text{Mat}_{2^n \times 2^n}(\mathbb{C})$.

For $A \in \mathcal{P}$, if $n \geq 1$ is large enough, let $A_n \in \mathcal{P}_n$ be defined the following way.

- $A_n(x, y) = A(x, y)$ if $2^n l \leq x, y \leq 2^n l + 2^n - 1$ for some $l \in \mathbb{Z}$.
- Otherwise, $A(x, y) = 0$.

Lemma 5.1. (i): $\{A_n\}_{n=1}^{\infty}$ is a Cauchy-sequence in \mathcal{M} .

(ii): $(A + B)_n = A_n + B_n$.

(iii): $\text{rk}_{\mathcal{M}}(A_n^* - (A^*)_n) = 0$.

(iv): $\text{rk}_{\mathcal{M}}((AB_n) - A_n B_n) = 0$.

(v): $\lim_{n \rightarrow \infty} A_n = 0$ if and only if $A = 0$.

Proof. First observe that for any $Q \in \mathcal{P}_n$

$$\text{rk}_{\mathcal{M}}(Q) \leq \frac{|\{0 \leq x \leq 2^n - 1 \mid \exists 0 \leq y \leq 2^n - 1 \text{ such that } A_n(x, y) \neq 0.\}|}{2^n}$$

Suppose that $A(x, y) = A(x + 2^k, y + 2^k)$ and $k < n < m$. Then

$$|\{0 \leq x \leq 2^n - 1 \mid A_n(x, y) \neq A_m(x, y) \text{ for some } 0 \leq y \leq 2^n - 1\}| \leq 2^k 2^{m-n}.$$

Hence by the previous observation, $\{A_n\}_{n=1}^{\infty}$ is a Cauchy-sequence. Note that (iii) and (iv) can be proved similarly, the proof of (ii) is straightforward. In order to prove (v) let us suppose that $A(x, y) = 0$ whenever $|x - y| \geq 2^k$. Let $n > k$ and $0 \leq y \leq 2^k - 1$ such that $A(x, y) \neq 0$ for some $-2^k \leq x \leq 2^k - 1$. Therefore $\text{rk}_{\mathcal{M}} A_n \geq \frac{2^{n-k}-1}{2^n}$. Thus (v) follows. \square
Let us define $\phi : \mathcal{P} \rightarrow \mathcal{M}$ by $\phi(A) = \lim_{n \rightarrow \infty} A_n$. By the previous lemma, ϕ is an injective $*$ -homomorphism. \square

Definition 5.2. A periodic operator A is diagonal if $A(x, y) = 0$, whenever $x \neq y$. The diagonal operators form the Abelian $*$ -algebra $\mathcal{D} \subset \mathcal{P}$.

Lemma 5.2. We have the isomorphism $\mathcal{D} \cong \mathbb{C}(\mathbb{Z}_{(2)})$, where $\mathbb{Z}_{(2)}$ is the Prüfer 2-group.

Proof. For $n \geq 1$ and $0 \leq l \leq 2^n - 1$ let $E_n^l \in \mathcal{D}$ be defined by

$$E_n^l(x, x) := \exp\left(\frac{2\pi i x (\text{mod } 2^n)}{2^n} l\right).$$

It is easy to see that $E_{n+1}^{2l} = E_n^l$ and the multiplicative group generated by E_n^1 is isomorphic to \mathbb{Z}_{2^n} . Observe that the set $\{E_n^l\}_{n,l,(l,n)=1}$ form a basis in the space of n -type diagonal operators. Therefore, $\mathcal{D} \cong \cup_{n=1}^{\infty} \mathbb{C}(Z_{2^n}) = \mathbb{C}(\mathbb{Z}_{(2)})$. \square

Let $J \in \mathcal{P}$ be the following element.

- $J(x, y) = 1$, if $y = x + 1$.
- Otherwise, $J(x, y) = 0$.

Then

$$(3) \quad J \cdot E_n^l = E_n^{l+1(\text{mod } 2^n)}.$$

Also, any periodic operator A can be written in a unique way as a finite sum

$$\sum_{k \in \mathbb{Z}} D_k \cdot J^k,$$

where D_k is a diagonal operator in the form

$$D_k = \sum_{n=0}^{\infty} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l.$$

Thus, by (2) and (3), we have the following corollary.

Corollary 5.1. *The map $\psi : \mathcal{P} \rightarrow \mathcal{A}_M \rtimes \mathbb{Z}$ defined by*

$$\psi\left(\sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{l,n,k} E_n^l \cdot k\right) = \sum_k \sum_{n \geq 0} \sum_{l|(l,n)=1} c_{l,n,k} F_n^l \cdot k$$

is a $$ -isomorphism of algebras.*

6. LÜCK'S APPROXIMATION THEOREM REVISITED

The goal of this section is to prove the following proposition.

Proposition 6.1. *We have $c(\rho) \cong \mathcal{M}$ where ρ is the odometer action.*

Proof. Let us define the linear map $t : \mathcal{P} \rightarrow \mathbb{C}$ by

$$t(A) := \frac{\sum_{i=0}^{2^n-1} A(i, i)}{2^n},$$

where $A \in \mathcal{P}$ and $A(x + 2^n, y + 2^n)$ for all $x, y \in \mathbb{Z}$.

Lemma 6.1. *$\text{Tr}_{\mathcal{N}(\rho)}(\psi(A)) = t(A)$, where ψ is the $*$ -isomorphism of Corollary 5.1.*

Proof. Recall that $\text{Tr}_{\mathcal{N}(\rho)}(F_n^l) = 0$, except, when $l = 0, n = 0, F_n^l = 1$. If $n \neq 0$ and $l \neq 0$, then $t(E_n^l)$ is the sum of all k -th roots of unity for a certain k , hence $t(E_n^l) = 0$. Also, $t(1) = 1$. Thus, the lemma follows. \square

It is enough to prove that

$$(4) \quad \text{rk}_{\mathcal{M}}(A) = \text{rk}_{\mathcal{N}(\rho)}(\psi(A))$$

Indeed by (4), ψ is a rank-preserving $*$ -isomorphism between \mathcal{P} and $\mathcal{A}_M \rtimes \mathbb{Z}$. Hence the isomorphism ψ extends to a metric isomorphism

$$\hat{\psi} : \overline{\mathcal{P}} \rightarrow \overline{\mathcal{A}_M \rtimes \mathbb{Z}},$$

where $\overline{\mathcal{P}}$ is the closure of \mathcal{P} in \mathcal{M} and $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is the closure of $\mathcal{A}_M \rtimes \mathbb{Z}$ in $U(\mathcal{N}(\rho))$. Since \mathcal{P} is dense in \mathcal{M} , $\overline{\mathcal{P}} \cong \mathcal{M}$. Also, $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is a $*$ -subalgebra of $U(\mathcal{N}(\rho))$, since the $*$ -ring operations are continuous with respect to the rank metric. Therefore $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is a continuous algebra isomorphic to \mathcal{M} . Observe that the rank closure $\overline{\mathcal{A}_M \rtimes \mathbb{Z}}$ is isomorphic to the rank closure of $L_c^\infty(\hat{\mathbb{Z}}_{(2)}, \mu_{\text{haar}}) \rtimes \mathbb{Z}$ by the argument of Lemma 3.2. Therefore, $c(\rho) \cong \mathcal{M}$. Thus from now on, our only goal is to prove (4).

Lemma 6.2. *Let $A \in \mathcal{P}$ and $A_n \in \text{Mat}_{2^n \times 2^n}(\mathbb{C})$ as in Section 5. Then the matrices $\{A_n\}_{n=1}^\infty$ have uniformly bounded norms.*

Proof. Let M, N be chosen in such a way that

- $|A_n(x, y)| \leq M$ for any $x, y \in \mathbb{Z}, n \geq 1$.
- $|A_n(x, y)| = 0$ if $|x - y| \geq \frac{N}{2}$.

Now let $v = (v(1), v(2), \dots, v(2^n)) \in \mathbb{C}^{2^n}$, $\|v\|^2 = 1$. Then

$$\begin{aligned} \|A_n v\|^2 &= \sum_{x=1}^{2^n} \left| \sum_{y: |x-y| < N/2} A_n(x, y) v(y) \right|^2 \leq M^2 \sum_{x=1}^{2^n} \left| \sum_{y: |x-y| < N/2} v(y) \right|^2 \leq \\ &\leq M^2 N \sum_{x=1}^{2^n} \sum_{y: |x-y| < N/2} |v(y)|^2 \leq M^2 \sum_{y=1}^{2^n} N |v(y)|^2 = M^2 N^2. \end{aligned}$$

Therefore, for any $n \geq 1$, $\|A_n\| \leq MN$. □

Lemma 6.3. *Let $A \in \mathcal{P}$. Then for any $k \geq 1$*

$$\lim_{k \rightarrow \infty} t((A_n^* A_n)^k) = t((A^* A)^k) = \text{Tr}_{\mathcal{N}(\rho)}(\psi(A^* A)^k).$$

Proof. Let $m \geq 1, l \geq 1, q \geq 1$ be integers such that

- $A(x, y) = A(x + 2^m, y + 2^m)$ for any $x, y \in \mathbb{Z}$.
- $A(x, y) = 0$, if $|x - y| \geq l$.
- $|(A^* A)^k(x, x)| \leq q$ and $|(A_n^* A_n)^k(x, x)| \leq q$ for any $x \in \mathbb{Z}$.

By definition,

$$\begin{aligned} t((A_n^* A_n)^k) &= \frac{\sum_{x=1}^{2^n} (A_n^* A_n)^k(x, x)}{2^n} \\ t((A^* A)^k) &= \frac{\sum_{x=1}^{2^n} (A^* A)^k(x, x)}{2^n}. \end{aligned}$$

Observe that if $2lk < x, 2^n - 2lk$, then

$$(A^* A)^k(x, x) = (A_n^* A_n)^k(x, x).$$

Hence,

$$|t((A^* A)^k) - t((A_n^* A_n)^k)| \leq \frac{4klq}{2^n}.$$

Thus our lemma follows. □

Now, we follow the idea of Lück [10]. Let μ be the spectral measure of $\psi(A) \in \mathcal{N}(\rho)$. That is

$$\text{Tr}_{\mathcal{N}(\rho)} f(A^* A) = \int_0^K f(x) d\mu(x),$$

for all $f \in C[0, K]$, where $K > 0$ is chosen in such a way that $\text{Spec } \psi(A^* A) \subset [0, K]$ and $\|A_n^* A_n\| \leq K$ for all $n \geq 1$. Also, let μ_n be the spectral measure of $A_n^* A_n$, that is,

$$t(f(A_n^* A_n)) = \int_0^K f(x) d\mu_n(x),$$

or all $f \in C[0, K]$. As in [10], we can see that the measures $\{\mu_n\}_{n=1}^\infty$ converge weakly to μ . Indeed by Lemma 6.3,

$$\lim_{n \rightarrow \infty} t(P(A_n^* A_n)) = \text{Tr}_{\mathcal{N}(\rho)} P(A^* A)$$

for any real polynomial P , therefore

$$\lim_{n \rightarrow \infty} t(f(A_n^* A_n)) = \text{Tr}_{\mathcal{N}(\rho)} f(A^* A)$$

for all $f \in C[0, K]$.

Since $\text{rk}_{\mathcal{M}}(A_n) = \text{rk}_{\mathcal{M}}(A_n^* A_n)$ and $\text{rk}_{\mathcal{N}(\rho)}(\psi(A)) = \text{rk}_{\mathcal{N}(\rho)}(\psi(A^* A))$, in order to prove (4) it is enough to see that

$$\lim_{n \rightarrow \infty} \text{rk}_{\mathcal{M}}(A_n^* A_n) = \text{rk}_{\mathcal{N}(\rho)}(\psi(A^* A)).$$

Observe that $\text{rk}_{\mathcal{M}}(A_n^* A_n) = 1 - \mu_n(0)$ and

$$\text{rk}_{\mathcal{N}(\rho)}(\psi(A^* A)) = 1 - \lim_{\lambda \rightarrow 0} \text{Tr}_{\mathcal{N}(\rho)} E_\lambda = \mu(0).$$

Hence, our proposition follows from the lemma below (an analogue of Lück's Approximation Theorem).

Lemma 6.4. $\lim_{n \rightarrow \infty} \mu_n(0) = \mu(0)$.

Proof. Let $F_n(\lambda) = \int_0^\lambda \mu_n(t) dt$ and $F(\lambda) = \int_0^\lambda \mu(t) dt$ be the distribution functions of our spectral measures. Since $\{\mu_n\}_{n=1}^\infty$ weakly converges to the measure μ , it is enough to show that $\{F_n\}_{n=1}^\infty$ converges uniformly. Let $n \leq m$ and $D_m^n : \text{Mat}_{2^n \times 2^n}(\mathbb{C}) \rightarrow \text{Mat}_{2^m \times 2^m}(\mathbb{C})$ be the diagonal operator. Let $\varepsilon > 0$. By Lemma 5.1, if n, m are large enough,

$$\text{Rank}(D_m^n(A_n) - A_m) \leq \varepsilon 2^m.$$

Hence, by Lemma 3.5 [2],

$$\|F_n - F_m\|_\infty \leq \varepsilon.$$

Therefore, $\{F_n\}_{n=1}^\infty$ converges uniformly. □

7. ORBIT EQUIVALENCE

First let us recall the notion of orbit equivalence. Let $\tau_1 : \Gamma_1 \curvearrowright (X, \mu)$ resp. $\tau_2 : \Gamma_2 \curvearrowright (Y, \nu)$ be essentially free probability measure preserving actions of the countably infinite groups Γ_1 resp. Γ_2 . The two actions are called orbit equivalent if there exists a measure preserving bijection $\Psi : (X, \mu) \rightarrow (Y, \nu)$ such that for almost all $x \in X$ and $\gamma \in \Gamma_1$ there exists $\gamma_x \in \Gamma_2$ such that

$$\tau_2(\gamma_x)(\Psi(x)) = \Psi(\tau_1(\gamma)(x)).$$

Feldman and Moore [4] proved that if τ_1 and τ_2 are orbit equivalent then $\mathcal{N}(\tau_1) \cong \mathcal{N}(\tau_2)$. The goal of this section is to prove the following proposition.

Proposition 7.1. *If τ_1 and τ_2 are orbit equivalent actions, then $c(\tau_1) \cong c(\tau_2)$.*

Our Theorem 1 follows from the proposition. Indeed, by Proposition 3.2 and Proposition 6.1

$$\mathcal{M} \cong c(\rho) \quad \text{and} \quad c(\mathbb{Z}_2 \wr H) \cong c(\tau_H).$$

By the famous theorem of Ornstein and Weiss [11], the odometer action and the Bernoulli shift action of a countably infinite amenable group are orbit equivalent. Hence $\mathcal{M} \cong c(\mathbb{Z}_2 \wr H)$. \square

Proof. We build the proof of our proposition on the original proof of Feldman and Moore. Let $\gamma \in \Gamma_1$, $\delta \in \Gamma_2$. Let

$$M(\delta, \gamma) = \{y \in Y \mid \tau_2(\delta)(y) = \Psi(\tau_1(\gamma)\Psi^{-1}(y))\}$$

$$N(\gamma, \delta) = \{x \in X \mid \tau_1(\gamma)(x) = \Psi^{-1}(\tau_2(\delta)\Psi(x))\}.$$

Observe that $\Psi(N(\delta, \gamma)) = M(\gamma, \delta)$. Following Feldman and Moore ([4], Proposition 2.1) for any $\gamma \in \Gamma_1$, $\delta \in \Gamma_2$

$$\kappa(\gamma) = \sum_{h \in \Gamma_2} h \cdot 1_{M(h, \gamma)}$$

and

$$\lambda(\delta) = \sum_{g \in \Gamma_1} g \cdot 1_{N(g, \delta)}$$

are well-defined. That is, $\sum_{n=1}^k h_n \cdot 1_{M(h_n, \gamma)}$ converges weakly to $\kappa(\gamma) \in \mathcal{N}(\tau_2)$ as $k \rightarrow \infty$ and $\sum_{n=1}^k g_n \cdot 1_{N(g_n, \delta)}$ converges weakly to $\lambda(\delta) \in \mathcal{N}(\tau_1)$ as $k \rightarrow \infty$, where $\{\gamma_n\}_{n=1}^\infty$ resp. $\{\delta_n\}_{n=1}^\infty$ are enumerations of the elements of Γ_1 resp. Γ_2 .

Furthermore, one can extend κ resp. λ to maps

$$\kappa' : L^\infty((X, \mu) \rtimes \Gamma_1) \rightarrow \mathcal{N}(\tau_2)$$

resp.

$$\lambda' : L^\infty((Y, \nu) \rtimes \Gamma_2) \rightarrow \mathcal{N}(\tau_1)$$

by

$$\kappa' \left(\sum_{\gamma \in \Gamma_1} a_\gamma \cdot \gamma \right) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \kappa(\gamma) = \sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^\infty h_n \cdot 1_{M(h_n, \gamma)}$$

and

$$\lambda' \left(\sum_{\delta \in \Gamma_2} b_\delta \cdot \delta \right) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \lambda(\delta) = \sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^\infty g_n \cdot 1_{N(g_n, \delta)}.$$

The maps κ' resp. λ' are injective trace-preserving $*$ -homomorphisms with weakly dense ranges. Hence they extend to isomorphisms of von Neumann algebras

$$\hat{\kappa} : \mathcal{N}(\tau_1) \rightarrow \mathcal{N}(\tau_2), \hat{\lambda} : \mathcal{N}(\tau_2) \rightarrow \mathcal{N}(\tau_1),$$

where $\hat{\kappa}$ and $\hat{\lambda}$ are, in fact, the inverses of each other.

Lemma 7.1.

$$(5) \quad \lim_{k \rightarrow \infty} \text{rk}_{\mathcal{N}(\tau_2)} \left(\sum_{\gamma \in \Gamma_1} (a_\gamma \circ \Psi^{-1}) \cdot \sum_{n=1}^k h_n \cdot 1_{M(h_n, \gamma)} - \hat{\kappa} \left(\sum_{\gamma \in \Gamma_1} a_\gamma \cdot \gamma \right) \right) = 0.$$

$$(6) \quad \lim_{k \rightarrow \infty} \text{rk}_{\mathcal{N}(\tau_1)} \left(\sum_{\delta \in \Gamma_2} (b_\delta \circ \Psi) \cdot \sum_{n=1}^k g_n \cdot 1_{N(g_n, \delta)} - \hat{\lambda} \left(\sum_{\delta \in \Gamma_2} b_\delta \cdot \delta \right) \right) = 0.$$

Proof. By definition, the disjoint union $\cup_{n=1}^\infty M(h_n, \gamma)$ equals to Y (modulo a set of measure zero). We need to show that if $\{\sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}\}_{k=1}^\infty$ weakly converges to an element $S \in \mathcal{N}(\tau_2)$, then $\{\sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}\}_{k=1}^\infty$ converges to S in the rank metric as well, where $T_n \in L_c^\infty(Y, \nu) \rtimes \Gamma_2$. Let $P_k = \sum_{n=1}^k 1_{M(h_n, \gamma)} \in l^2(\Gamma, L^2(Y, \nu))$. We denote by \hat{P}_k the element $\sum_{n=1}^k 1_{M(h_n, \gamma)}$ in $L_c^\infty(Y, \nu) \rtimes \Gamma_2$. By definition, if $L(A)(P_k) = 0$ then $A\hat{P}_k = 0$. Now, by weak convergence,

$$L(S)(P_k) = \lim_{l \rightarrow \infty} \sum_{n=1}^l T_n \cdot 1_{M(h_n, \gamma)}(P_k).$$

That is,

$$L(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})(P_k) = 0.$$

Therefore,

$$(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})\hat{P}_k = 0.$$

Thus,

$$(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}) = (S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)})(1 - \hat{P}_k).$$

By Lemma 3.1, $\text{rk}_{\mathcal{N}(\tau_2)}(1 - \hat{P}_k) = 1 - \sum_{n=1}^k \nu(M(h_n, \gamma))$, hence

$$\lim_{k \rightarrow \infty} \text{rk}_{\mathcal{N}(\tau_2)}(S - \sum_{n=1}^k T_n \cdot 1_{M(h_n, \gamma)}) = 0. \quad \square$$

Now let us turn back to the proof of our proposition. By (5), $\hat{\kappa}$ maps the algebra $L_c^\infty(X, \mu) \rtimes \Gamma_1$ into the rank closure of $L_c^\infty(Y, \nu) \rtimes \Gamma_2$. Since $\hat{\kappa}$ preserves the rank, $\hat{\kappa}$ maps the rank closure of $L_c^\infty(X, \mu) \rtimes \Gamma_1$ into the rank closure of $L_c^\infty(Y, \nu) \rtimes \Gamma_2$. Similarly, $\hat{\lambda}$ maps the rank closure of $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ into the rank closure of $L_c^\infty(X, \mu) \rtimes \Gamma_1$. That is, $\hat{\kappa}$ provides an isomorphism between the rank closures of $L_c^\infty(X, \mu) \rtimes \Gamma_1$ and $L_c^\infty(Y, \nu) \rtimes \Gamma_2$. Therefore, the smallest continuous ring containing $L_c^\infty(X, \mu) \rtimes \Gamma_1$ in $U(\mathcal{N}(\tau_1))$ is mapped to the smallest continuous ring containing $L_c^\infty(Y, \nu) \rtimes \Gamma_2$ in $U(\mathcal{N}(\tau_2))$. \square

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